Dynamics of modulated waves in electrical lines with dissipative elements

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By a means of a method based on the reductive perturbation method, we show that the amplitude of waves on the nonlinear electrical transmission lines (NLTLs) is described by the cubic-quintic complex Ginzburg-Landau (CGL) equation. Then, we revisit analytically and numerically the processes of modulational instability (MI). The evolution of dissipative modulated waves through the network is also examined, and we show that solitonlike excitations can be induced by MI. Analytical results, illustrating the nature of MI of plane-wave solution, are also found to be in good agreement with numerical findings.

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Solitonlike localized states arise in dissipative systems driven far from thermal equilibrium such as hydrodynamics [1], granular media [2], Bose-Einstein condensates [3], nonlinear optics [4], and nonlinear electrical transmission line [5], to cite a few. These structures are referred to as "dissipative solitons" and are sustained because of an interplay between dispersion, nonlinearity, gain, and losses. The term "dissipative," in its present meaning, was introduced in the work of Nicolis and Prigogine [6] to describe systems which have losses as well as a pump source. One of the models of a dissipative system is based on the complex Ginzburg-Landau (CGL) equation [7,8] that has terms responsible for a variety of gain-loss mechanisms. This equation also describes self-phase modulation, as well as diffraction which manifests itself as a discriminator for various angular components of wave. One of its experimental realizations is spatial dissipative solitons in semiconductors [9].

Nonlinear electrical transmission lines (NLTLs) are very convenient tools for studying wave propagation in nonlinear dispersive media [5,10]. The nonlinear propagation of signals in NLTLs has been investigated theoretically and numerically by many authors [11]. It has been shown that the system of equations governing the physics of this network can be reduced to a cubic nonlinear Schrödinger (NLS) equation or a pair of coupled nonlinear Schrödinger (CNLS) equations, the Korteweg–de Vries equation, and the CGL equation. Modulational instability (MI) is the outcome of the interplay between nonlinearity and dispersive or diffraction effects. It is a symmetry-breaking instability so that a small perturbation on top of a constant-amplitude background experiences exponential growth, and this leads to beam breakup in either space or time.

In this Brief Report, we analyze the dynamics of waves in a nonlinear electrical line with driving and dissipation. We focus on the derivation of the cubic-quintic (CQ) complex Ginzburg-Landau (CQCGL) equation from the discrete equation, and study the propagation of modulated waves as well as effects of dissipative elements induced by MI.

We consider a nonlinear network of N cells as illustrated in Fig. 1. Each cell contains a linear inductance L_1 in the series branch and a linear inductance L_2 in parallel with a nonlinear capacitance $C(V)$ in the shunt branch. This capacitance consists of a reverse-biased diode with a differential capacitance function of the voltage V_n across the nth capacitor. In order to take into account the dissipation through the network, the conductances g_1 and g_2 are connected in parallel with L_1 and L_2, respectively. The conductance g_1 describes the dissipation in the inductance L_1, while g_2 accounts for the dissipation of the inductance L_2 in addition to the loss of the nonlinear capacitance C_V. The nonlinearity is introduced in the line by a Varicap diode for which the capacitance varies with the applied tension. Its capacitance $C(V_n)=C_0(1-\alpha V_n+\beta V_n^2)$ depends nonlinearly on the voltage V_n of the nth cell, with positive parameters C_0, α, and β. From Kirchhoff's laws it is easy to show that the propagation of waves in the network is governed by the following equation:

$$\frac{d^2V_n}{dt^2}-\alpha\frac{d^2V_n^2}{dt^2}=\mu_0^2(V_{n-1}-2V_n+V_{n+1})+2\mu_0\sigma_1\left(\frac{d(V_{n-1}-2V_n+V_{n+1})}{dt}\right)-\omega_0^2V_n-2\mu_0\sigma_2\frac{dV_n}{dt}, \qquad (1)$$

with $n=1,2,\ldots,N$, where N is the number of cells considered. Coefficients $\mu_0=1/(L_1C_0)^{1/2}$ and $\omega_0=1/(L_2C_0)^{1/2}$ are characteristic frequencies of the network. For the sake of convenience, the dimensionless σ_1 and σ_2 are introduced, and they are related to conductances g_1 and g_2 as $\frac{g_1}{C_0}=2\mu_0\sigma_1$ and $\frac{g_2}{C_0}=2\mu_0\sigma_2$. We use the semidiscrete approxi-

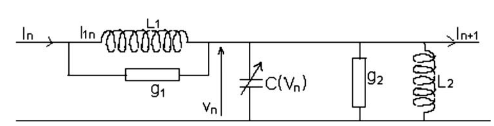

FIG. 1. One unit cell of the discrete nonlinear electrical transmission line.

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mation to obtain the short-wavelength envelope soliton. This asymptotic approach allows us to describe the envelope in the continuum approximation and to treat properly the carrier wave with its discrete character. Owing to the assumed weak nonlinearity, we expand $V_n(t)$ into the following asymptotic series [12]:

$$V_n(t) = \sum [\varepsilon^{l/2} V_{l,m}(n,t)] e^{[im\theta(n,t)]} + \text{c.c.} \tag{2}$$

Utilizing the idea developed by Taniuti and Yajima [13], the solution $V_n(t)$ is taken to be

$$\begin{aligned} V_n(t) = {} & \varepsilon^{1/2} V_{11}(n,t) e^{i\theta} + \text{c.c.} + \varepsilon[V_{20} + V_{22}(n,t)e^{2i\theta} + \text{c.c.}] \\ & + \varepsilon^{3/2} V_{33}(n,t) e^{3i\theta} + \text{c.c.} + \varepsilon^2 [V_{42}(n,t)e^{2i\theta} \\ & + V_{44}(n,t)e^{4i\theta} + \text{c.c.}] + \varepsilon^{5/2}[V_{53}(n,t)e^{3i\theta} \\ & + V_{55}(n,t)e^{5i\theta} + \text{c.c.}] + 0(\varepsilon^{7/2}), \end{aligned} \tag{3}$$

where θ is the phase defined by $\theta = kn - \omega t$. The smallness parameter ε which ranks from 0 to 1 $(0 < \varepsilon \leqslant 1)$ represents the size of the amplitude of perturbation; c.c. stands for the complex conjugate of the preceding expression. In the resulting equation, there are nonzero terms $V_{lm}(n \pm 1)$, which are expanded in the continuum approximation around $V_{lm}(x,t)$, with $n = x$. So, the fast change in the phase θ in Eq. (3) is correctly taken into account by considering differences in the phase for the discrete variable n. We have also scaled time and space derivatives as $\frac{\partial}{\partial x} \sim 0(\varepsilon)$ and $\frac{\partial}{\partial t} \sim 0(\varepsilon)$, respectively, and neglected consistently high order in ε terms. Then, we keep up to the second derivative terms of $V_n(t)$ to balance dispersion and nonlinearity. Substitution of $V_n(t)$ and its derivatives in Eq. (1) yields a series of equations with respect to the power of ε.

From equations of $(\varepsilon^{1/2}, e^{i\theta})$, that is, the terms of $0(\varepsilon)^{1/2}$ for the first harmonic, we obtain the linear dispersion relation

$$\omega^2 = \omega_0^2 + 4\mu_0^2 \sin^2\left(\frac{k}{2}\right). \tag{4}$$

From the equation of $(\varepsilon^3, e^{0i\theta})$, we obtain the expression of the group velocity V_g defined by

$$V_g = \frac{\partial \omega}{\partial k} = \frac{4\mu_0^2 \sin(k)}{\omega}, \tag{5}$$

which is represented in Fig. 2. At the order of $(\varepsilon^{3/2}, e^{i\theta})$, we have

$$\begin{aligned} & -2i\omega \frac{\partial V_{11}}{\partial t} + 4\mu_0 \sin(k)(i\mu_0 + \omega\sigma_1)\frac{\partial V_{11}}{\partial x} \\ & \quad = -2\alpha\omega^2[(a_1 + 2\chi_{20}) + ia_2]|V_{11}|^2 V_{11}, \end{aligned} \tag{6}$$

where $\chi_{20} = \frac{2\alpha V_g^2}{V_g^2 - \mu_0^2}$. Therefore at $(\varepsilon^{5/2}, e^{3i\theta})$, we can write

$$\begin{aligned} V_{53} = {} & (a_{31} + ia_{32})\frac{\partial (V_{11}^3)}{\partial x} + (a_{33} + ia_{34}) V_{11} \frac{\partial (V_{11}^2)}{\partial x} \\ & + (a_{35} + ia_{36})|V_{11}|^2 V_{11}^3, \end{aligned} \tag{7}$$

where coefficients a_j and a_{ij} are defined in the Appendix. From the equation of $(\varepsilon^{5/2}, e^{i\theta})$ and by going into the reference frame moving with the group velocity, the resulting equation describing the dynamics of a wave packet has the following form:

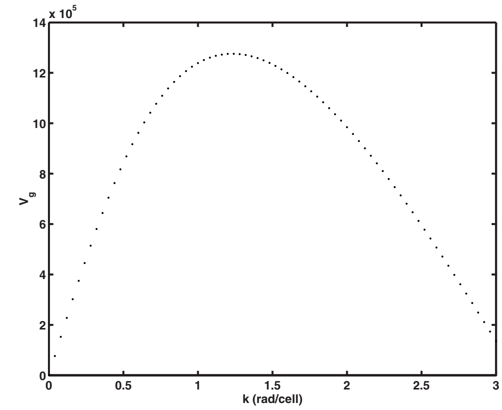

FIG. 2. Representation of the group velocity as a function of the wave number k.

$$iV_{11,\tau}(\tau) + P\frac{\partial^2 V_{11}}{\partial \xi^2} + Q_1|V_{11}|^2 V_{11} + Q_2 |V_{11}|^4 V_{11} = 0, \tag{8}$$

where $\xi = x - V_g t$, $\tau = t$, and the subscripts τ and ξ denote partial differentiations with respect to τ and ξ. Coefficients $P = (P_r + iP_i)$ and $Q_j = (Q_{jr} + iQ_{ji})$ [with $j = 1, 2$] are expressed in the Appendix. P_r is the dispersion coefficient; P_i describes spectral filtering or parabolic gain. Q_{1r} determines how the frequency is amplitude modulated, while Q_{1i} accounts for the cubic nonlinear amplification. Q_{2r} represents quintic nonlinearity and Q_{2i} is the quintic damping.

The cubic CGL equation gives rise to exact solutions for solitary pulses, but they are unstable. The most straightforward way to modify the equation so as to provide for the existence of stable pulses is to introduce the CQ nonlinearity. Equation (8) is the so-called CQCGL equation. The CQCGL equation was originally proposed by Petviashvili and Sergee [14], and stable solitary pulses in this model were first predicted in Ref. [15]. Later, solutions of the CQCGL equation were investigated in great detail [15]. Among other physical applications of the CQCGL equation, one can also mention binary fluid convection [16], phase transition [17], Taylor-Couette flow between counter-rotating cylinders [18], and soliton fiber laser with nonlinear polarization-dependent losses (which is equivalent to fast saturable absorption action) [19]. In this case, the time-localized pulse is supported by the nonlinear gain and loses energy. Thus, a stable stationary soliton state may be formed as a result of the balance between nonlinear gain, spectral filtering, and the quintic stabilizing term.

In order to study the MI, we consider a small perturbation of the initial wave as

$$V_{11}(\xi,\tau) = [1 + B(\xi,\tau)] A_n \exp[i(k_n\xi - \omega_n \tau)], \tag{9}$$

where the perturbation $B(\xi,\tau)$ is considered to be a combination of progressive and regressive waves, A_n is a complex constant, k_n and ω_n are the wave number and the angular frequency of the carrier wave, respectively, and l and Ω are the wave number and the angular frequency of the perturbation, respectively. Following the standard procedure of linear stability analysis presented in Ref. [20], one can derive the following MI criterion for the system understudy:

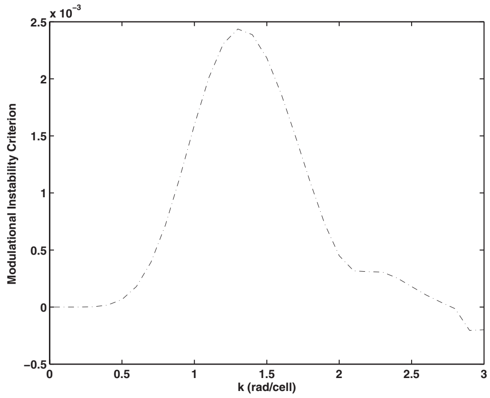

FIG. 3. Modulational instability criterion and regions of MI.

$$(P_rQ_{1r} + P_iQ_{1i}) + 2(P_rQ_{2r} + P_iQ_{2i})|A_n|^2 + \frac{\Gamma}{2l^2|A_n|^2} > 0, \quad (10)$$

where

$$\Gamma = -16Q_{2i}Q_{1i}|A_n|^6 - 8P_il^2Q_{2i}|A_n|^4. \quad (11)$$

Relation (10) represents the MI criterion associated to CQCGL equation (8). This result generalizes the well-known Lange and Newell criterion for Stokes waves by the presence of the additional term $\frac{\Gamma}{2l^2|A_n|^2}$. Figure 3 depicts the right-hand side of relation (10). From this figure one can see that dispersion relation (10) is positive for wave number in the range of $0<k<2.5$.

In order to check the validity of the analytical predictions on MI presented above, we have performed numerical simulations on the general Eq. (1) governing wave propagation in the NLTL. Parameters of the line are $L_1=0.640$ mH, $L_2=0.480$ mH, and $\alpha=0.21$ V^{-1}, with $\sigma_1=0.004$ and $\sigma_2=0.001$ [10,21]. The fourth-order Runge-Kutta scheme is used with a normalized integration time step $\Delta t=5\times10^{-3}$. Similarly, the number of cells is chosen so that we do not encounter the wave reflection at the end of the line. At the input of the line, we apply a slowly modulated signal:

$$V(t) = V_0[1 + m_0 \cos(2\pi f_m t)]\cos(2\pi f_p t), \quad (12)$$

where V_0 is the amplitude of the unperturbed plane wave (carrier wave), m_0 designates the modulation rate, and f_m is the frequency of modulation. As a specific example, we use the following values: $V_0=1.5$ V, $f_p=1180$ kHz, $m_0=0.01$, and $f_m=16$ kHz. Figure 4 shows an example of wave propagating through the network in the absence of dissipation terms ($\sigma_1=\sigma_2=0$). As the time goes on and as the wave travels along the electrical network, we observe the propagation of wave packet. The magnitude of wave decreases exponentially. If we take into account dissipation on the line ($\sigma_1=0.004$ and $\sigma_2=0.001$), we observe in Fig. 5 that the initial nonlinear excitation is well modulated. Waves propagate through the electrical network; the continuous wave breaks into a pulse train. The solitonic excitations of the pulse train have envelope functions with a familiar shape of the theory of solitonlike objects. Each element of the train has the shape of a solitonlike object. But, in contrast to soliton, they emerge as a solution of time-dependent classical equation of motion. In fact, the typical occurrence of soliton-

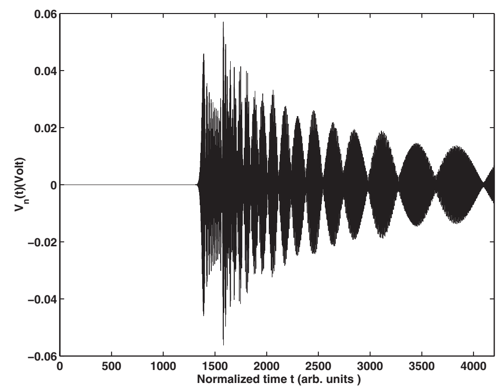

FIG. 4. Disintegration of plane wave into solitonlike excitation in the line in the absence of dissipative elements for cell 700.

like pulses (hereafter we call them soliton) produced by MI along the evolution of waves is due to the interplay between the nonlinearity and dispersion. The first experimental observation of MI was reported by Tai *et al.* [22] on the light waves in dielectric material.

By comparing Figs. 4 and 5, one can note that the magnitude of waves has drastically decreased due to the presence of the dissipative terms in the line. Since the disintegration on pulses train typically occurs in the same parameter region where bright solitons are observed, MI is considered, to some extent, a precursor to soliton formation. MI is then responsible for the formation of envelope soliton in electrical transmission lines. MI also sets a fundamental nonlinear limiting factor in the transmission of dense wavelength-division multiplexed signals in long-distance electrical links.

In summary, we have considered the discrete NLTL and examined the dynamics of modulated waves. Through the reductive perturbation method, it has been shown that the propagation of modulated waves is governed by the CQCGL equation. Based on this equation and exploiting the Stokes wave analysis, the generalized Lange and Newell criterion for MI has been derived. It has been found that when the line is subjected to MI, the initial waves disintegrate into a train of pulses. Excellent agreement between analytical and numerical investigations of MI has been obtained.

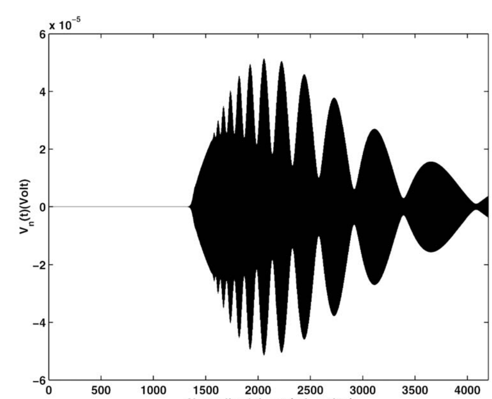

FIG. 5. Disintegration of plane wave into solitonlike excitation in the line in the presence of dissipation of dissipative elements for cell 700.

APPENDIX

The following are expressions of different coefficients a_i and a_{ij}:

$$a_0 = \omega_0^2 + 4\mu_0^2 \sin^2(k) - 4\omega^2,$$

$$a_{01} = -4\omega\mu_0[\sigma_2 + 4\sigma_1 \sin^2(k)],$$

$$a_1 = \frac{-2\alpha a_0}{a_0^2 + a_{01}^2}, \quad a_2 = \frac{-2\alpha a_{01}}{a_0^2 + a_{01}^2},$$

$$a_3 = \omega_0^2 + 4\mu_0^2 \sin^2\left(\tfrac{3k}{2}\right) - 9\omega^2,$$

$$a_4 = -6\omega\mu_0\left[\sigma_2 + 4\sigma_1 \sin^2\left(\tfrac{3k}{2}\right)\right],$$

$$a_5 = -\frac{6\alpha(a_1a_3 + a_2a_4)}{a_3^2 + a_4^2},$$

$$a_6 = -\frac{6\alpha(a_2a_3 - a_1a_4)}{a_3^2 + a_4^2}, \quad a_7 = a_0, \quad a_8 = a_{01},$$

$$a_{12} = -2\alpha\omega^2 a_6, \quad a_{13} = \frac{(a_7a_9 + a_8a_{10})}{a_7^2 + a_8^2},$$

$$a_9 = 2\mu_0 V_g a_1 - 8\omega\mu_0 V_g a_1\sigma_1 \sin(2k) + 2\mu_0^2 V_g a_2 \sin(2k),$$

$$a_{14} = \frac{(a_7a_{10} - a_8a_9)}{a_7^2 + a_8^2}, \quad a_{11} = -2\alpha\omega^2 a_5,$$

$$a_{10} = 4\alpha\omega\mu_0 V_g - 2\mu_0^2 V_g a_1\sigma_1 \sin(2k) + 2\mu_0 V_g \sigma_2 a_2 \sin(2k) - 8\omega\mu_0 V_g a_2\sigma_1 \sin(2k), \quad a_{23} = a_3,$$

$$a_{15} = \frac{(a_7a_{11} + a_8a_{12})}{a_7^2 + a_8^2}, \quad a_{16} = \frac{(a_7a_{12} - a_8a_{11})}{a_7^2 + a_8^2},$$

$$a_{17} = \omega_0^2 + 4\mu_0^2 \sin^2(2k) - 16\omega^2, \quad a_{24} = a_4,$$

$$a_{18} = -8\omega\mu_0[\sigma_2 + 4\sigma_1 \sin^2(2k)],$$

$$a_{19} = -32\alpha\omega^2 a_5 - 16\alpha\omega^2(a_2^2 - a_3^2),$$

$$a_{20} = -32\alpha\omega^2(a_6 - a_2a_3),$$

$$a_{21} = \frac{(a_{17}a_{19} + a_{18}a_{20})}{a_{17}^2 + a_{18}^2}, \quad a_{22} = \frac{(a_{17}a_{20} - a_{18}a_{19})}{a_{17}^2 + a_{18}^2},$$

$$a_{25} = -V_g\frac{a_4}{3}a_5 - 6\omega V_g(a_6 + 2a_2),$$

$$a_{27} = -6\alpha\omega^2 a_{13}, \quad a_{28} = -6\alpha\omega^2 a_{14},$$

$$a_{26} = -6\omega V_g(a_5 - 2a_1) - V_g\frac{a_4}{3}a_6, \quad a_{29} = a_5 + a_{15} + 2a_{21},$$

$$a_{30} = a_6 + a_{16} + 2a_{22}, \quad a_{31} = \frac{(a_{23}a_{25} + a_{24}a_{26})}{a_{23}^2 + a_{24}^2},$$

$$a_{32} = \frac{(a_{23}a_{26} - a_{24}a_{25})}{a_{23}^2 + a_{24}^2}, \quad a_{33} = \frac{(a_{23}a_{27} + a_{24}a_{28})}{a_{23}^2 + a_{24}^2},$$

$$a_{34} = \frac{(a_{23}a_{28} - a_{24}a_{27})}{a_{23}^2 + a_{24}^2}, \quad a_{35} = \frac{(a_{23}a_{29} + a_{24}a_{30})}{a_{23}^2 + a_{24}^2},$$

$$a_{36} = \frac{(a_{23}a_{30} - a_{24}a_{29})}{a_{23}^2 + a_{24}^2}, \quad P = (a_{45} + ia_{46})$$

$$Q_{1r} = -\frac{\frac{4\alpha^2\omega V_g^2}{V_g^2 - \mu_0^2} + a_1\alpha\omega}{\varepsilon}, \quad Q_{1i} = -\frac{\alpha\omega a_2}{\varepsilon},$$

$$Q_{2r} = -\alpha\omega(a_5a_1 + a_6a_2), \quad Q_{2i} = -\alpha\omega(a_6a_1 - a_5a_2).$$

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